# ECON-UN 3211 - Intermediate Microeconomics

## Recitation 5: The Producer’s Problem I - Cost Minimization

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# Logistics

- Midterm
  - Consumer theory recordings now uploaded
  - Today's recording will be posted by Saturday
  - In addition to annotated notes, will also post midterm review exercises not covered today
- No problem set this week
- Problem Set 6 and recitation as usual next week

# Today: reviewing producer theory up to cost minimization

Introduction to producer theory

The Producer's Problem I: Cost Minimization

- Properties of production technologies
- Solving the cost-minimization problem
- Examples of production technologies

Other cost functions

# Introduction to producer theory

# Modeling the producer: what's a producer?

For our purposes, the producer is a 'firm' making production decisions to maximize profit $\pi$. But what even is a firm?

- Governments, oil companies, startups, independent artists: can one model describe all of them?
- Even in a classic 'firm', whose interests are the firm's interests?
    - Corporate boards, middle managers, workers, consultants, even consumers
    - Principal-agent problems: do different people have conflicts of interest?
    - Structure: firm behavior when managers are owners? when workers are owners? when consumers are owners?
    - Dynamics: do objectives change? are they consistent in short and long term?
- Other possible objectives: maximize revenue, quantity produced, employment, producer surplus, stock price; minimize output price?

# The producer's problem

$$\max \pi = \max(\text{Revenue} - \text{Cost})$$
$$= \max_q \{p(q) \times q - c(q)\}$$

- Seems easy to solve
  - In a perfectly competitive market, profits are always equal to zero and revenue is equal to costs
  - But if markets are not perfectly competitive (which in real life is always), revenue and costs are generally not equal
  - A single choice variable $q$ with no apparent constraint?
- $p(q)$: the inverse demand function, comes from the "demand side"
- $c(q)$: the *minimized* cost of producing at quantity $q$
- So we break down the producer's problem into two parts

# Profit maximization as two sequential optimizations

$$\max_q p(q) \times q - c(q)$$

**I. Cost minimization**:

- taking as given a desired quantity of production $q$, prices $p$, and technology $f(x_1, x_2)$
- find the least-cost method of producing $q$ output using goods 1 and 2

**II. The supply decision**:

- given the cost function $c(q)$ derived in the first step
- choose the quantity of production $q^* \geq 0$ that maximizes profit

# The Producer's Problem I: Cost Minimization

# The Producer's Problem I: Cost Minimization

$$\max(\text{Profit}) = \max(\text{Revenue} - \text{Cost}) = \max_q \{p(q)q - c(q)\}$$

Step 1: Cost minimization

$$\min_{\{x_1, x_2\}} w_1 x_1 + w_2 x_2$$

$$\text{s.t. } f(x_1, x_2) \geq q$$

Given:

- technological constraint $f(x_1, x_2)$
- input prices $w_1, w_2$
- output quantity $q$

Derive:

- conditional factor demand functions

$$x_1^*(w_1, w_2, q)$$

$$x_2^*(w_1, w_2, q)$$

- cost function

$$c(q) = w_1 x_1^*(w_1, w_2, q) + w_2 x_2^*(w_1, w_2, q)$$

# Cost minimization: comparisons to consumer's problem

Preferences

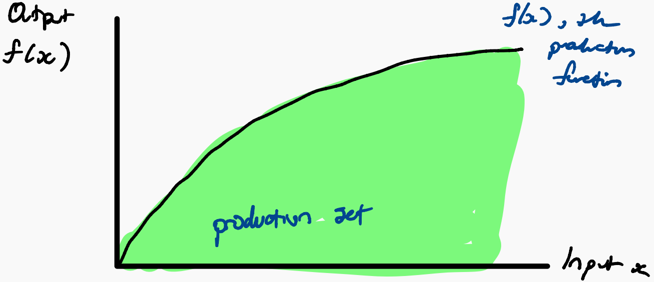
- Preferences: utility functions map bundles of goods $(x_1, x_2)$ to a level of utility
- Consumer compares how these goods contribute to their utility vs. how they are priced
- Evaluating these tradeoffs leads to optimal quantities
- Expenditure is the product of these good quantities and their prices
- Difference: utility is only ordinal

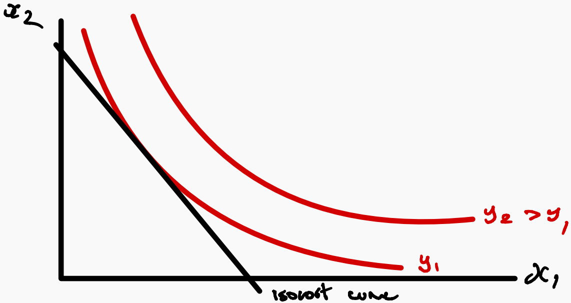
Technology

- Technology: production functions map bundles of inputs $(x_1, x_2)$ to a level of production
- Producer compares how these inputs contribute to their production vs. how they are priced
- Evaluating these tradeoffs leads to optimal input quantities
- Cost is the product of these input quantities and their prices
- Difference: production is "real", both ordinal and cardinal

# Technology as a constraint

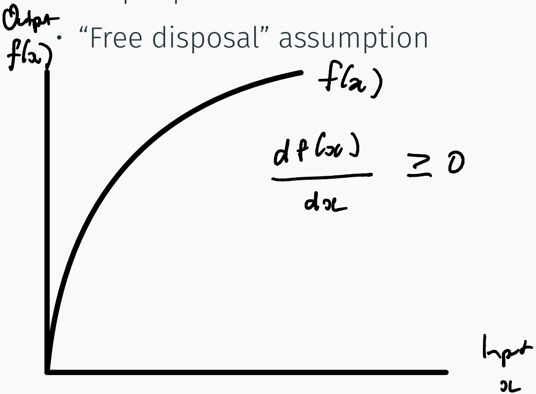
- Inputs or "factors of production" $(x_1, x_2)$ or $(L, K)$
  - Raw materials
  - Land
  - Labor
  - Physical capital
- Given a bundle of inputs $(x_1, x_2)$, there is a limit to what can be produced under the available technology
  - Figure: **production set/function** in single-input $y - x$ space, similar to budget set/line
  - Figure: **isoquant** in $x_2 - x_1$ space, similar to indifference curve



# Properties of production technologies: monotonicity and convexity

- More of any input will never reduce the amount you can produce
- Graph: production set
- "Free disposal" assumption




- Convexity: at least as efficient to produce using combinations of inputs as single inputs on their own
- Suppose there exist two ways of producing 1 unit of output
  1. Method A using $(a_1, a_2)$ inputs
  2. Method B using $(b_1, b_2)$ inputs
- Suppose we want to produce 100 units of output. We can use Method A 100 times or Method B 100 times
- Alternatively, any convex combination (e.g. 75 units under A, 25 units under B) could produce at least the same output with the same resources

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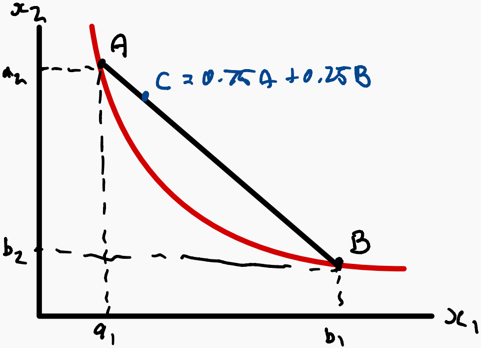


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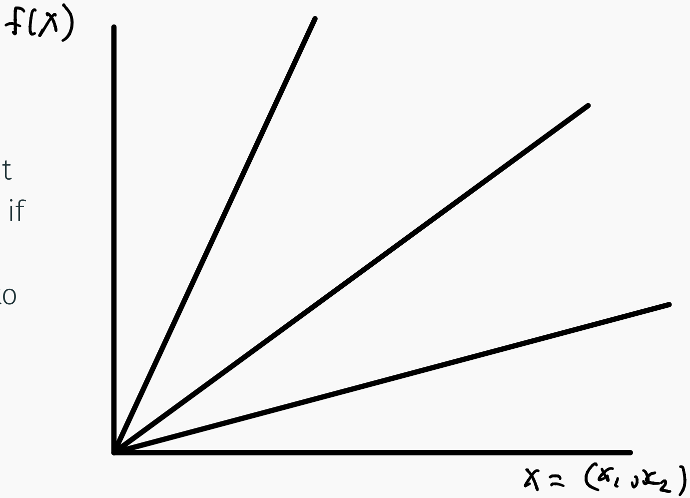
# Properties of production technologies: returns to scale

- Suppose we're initially producing $f(x_1, x_2)$
- What happens when we keep the mix of inputs the same, but just "scale" them up by doubling the inputs? Or tripling?
- Monotonicity assumption requires that output increases, but is this increase proportional? It depends on the production function $f(x_1, x_2)$
- This is a question we didn't address in consumer choice because utility is purely ordinal. Here, quantity produced is ordinal and cardinal so this is a pertinent question
- Mathematically, for some scaling factor $t > 1$:

$$\begin{aligned} f(tx_1, tx_2) &= tf(x_1, x_2) \\ &> tf(x_1, x_2) \\ &< tf(x_1, x_2) \end{aligned}$$

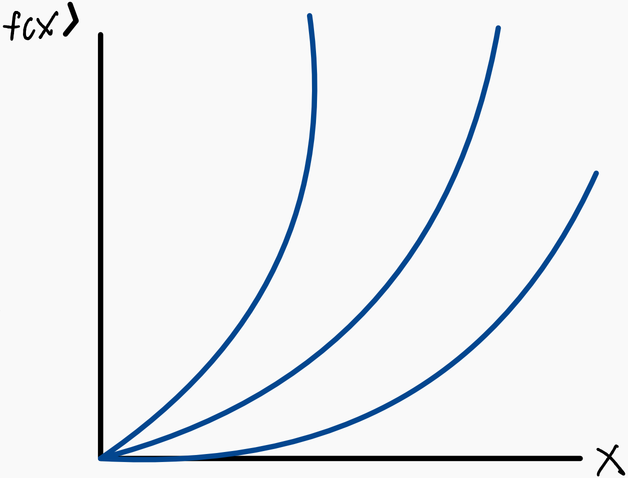
# Constant returns to scale: $f(tx_1, tx_2) = tf(x_1, x_2)$

- If we have a factory producing at $f(x_1, x_2)$, the technology has CRS if we can just build another three factories doing the same thing to quadruple production



# Increasing returns to scale: $f(tx_1, tx_2) > tf(x_1, x_2)$

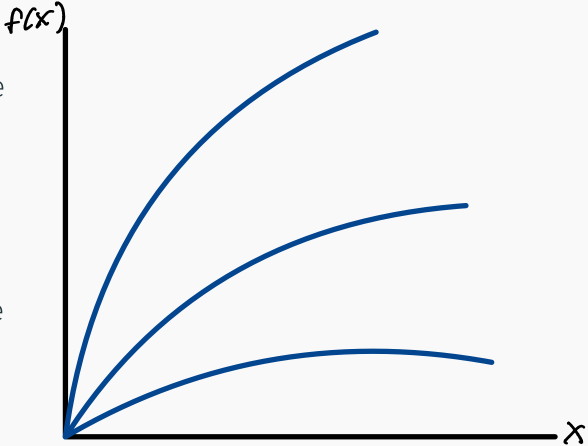
- When there are productivity benefits from scaling
- Example: Silicon Valley tech companies locate near each other for mutual benefit



# Decreasing returns to scale: $f(tx_1, tx_2) < tf(x_1, x_2)$

Non-textbook example: low-hanging fruit

- Imagine I'm one farmer working one acre of land to produce $f(1, 1)$
- I can buy another acre and hire another farmer to produce $f(2, 2)$
- But if the first plot of land I bought was the best piece of land available, then the next plot of land might have worse features (different geography, less fertile soil) and so it won't be as productive
- Then $f(2, 2) < 2f(1, 1)$



# Example 1: $f(x_1, x_2) = x_1^2 x_2^2$

$$f(tx) = (tx_1)^2 (tx_2)^2$$

$$= t^4 x_1^2 x_2^2$$

$$= t^4 f(x_1, x_2)$$

$$= t^3 t f(x_1, x_2) \geq t f(x_1, x_2) \text{ when } t \geq 1$$

<u>IRS</u>

# Example 2: $f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$

$$f(tx_1, tx_2) = A(tx_1)^{\alpha}(tx_2)^{\beta}$$

$$= A t^{\alpha+\beta} x_1^{\alpha} x_2^{\beta}$$

$$= t^{\alpha+\beta} A x_1^{\alpha} x_2^{\beta} \quad \text{vs.} \quad t A x_1^{\alpha} x_2^{\beta}$$

IRS if $\alpha+\beta > 1$

CRS if $\alpha+\beta = 1$

DRS if $\alpha+\beta < 1$

# Marginal product and the technical rate of substitution

Marginal product $MP_i(x_1, x_2)$

- The additional output that results from using an additional unit of input $i$, **keeping the other inputs fixed**
- Comparable to marginal utility (but with a direct interpretation)

Technical rate of substitution $TRS(x_1, x_2)$

- The amount of input 2 needed to offset the production lost by reducing input 2 by one unit
- Comparable to the marginal rate of substitution (tradeoff between goods that maintains the same level of utility)
- Equivalent to the slope of the isoquant (comparable to the slope of the indifference curve)

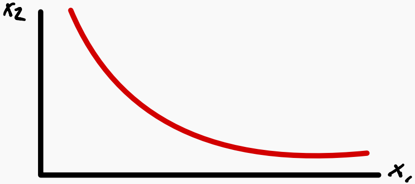
# Diminishing marginal product and diminishing technical rate of substitution

Diminishing MP

- We are looking at the production resulting from a one-unit increase in one input, holding all other inputs fixed (so no tradeoff)
- Example: farmer working in a large field (changing labor input, holding land input fixed)

Diminishing TRS

- Slope of isoquant decreases in magnitude as $x_1$ increases, increases as $x_2$ increases
- There is some intermediate value at which the marginal products of the two inputs are equal and so TRS is -1



# Solving the cost-minimization problem

$$\min_{\{x_1,x_2\}} w_1x_1 + w_2x_2$$
$$\text{s.t. } f(x_1, x_2) \geq q$$

- Interior solution for well-behaved technology comes from constraint binding + a tangency condition:

$$TRS(x_1, x_2) = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)} = -\frac{w_1}{w_2}$$

- The solution to the optimization problem is the conditional factor demand:

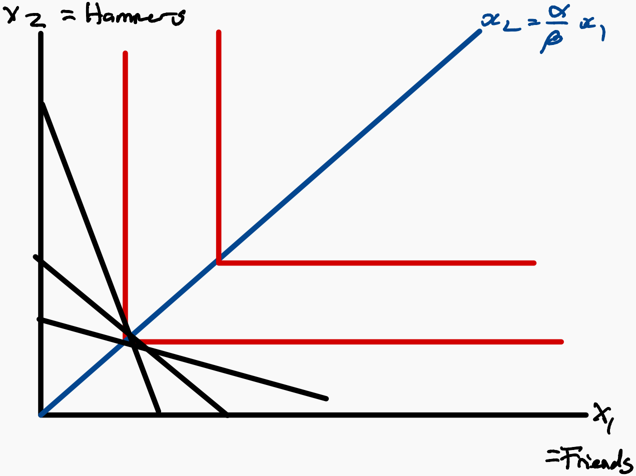
$$x^*(w_1, w_2, q)$$

“Conditional” because it is conditional on wanting to produce at some output level $q$, which may not be the optimal quantity $q^*$ to maximize profit

- Plug into the objective function to get the optimized cost function

# Fixed-proportions production: $f(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$

- Example: doing home improvement
  - Inputs: one person, one hammer
  - An additional hammer doesn't make one person more productive
  - An additional person doesn't make one hammer more productive
- Straightforward comparison to perfect complements in consumer theory



# Fixed-proportions production: $f(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$

- If we want to produce $q$ units of output, we need at least $\frac{q}{\alpha}$ units of $x_1$ and $\frac{q}{\beta}$ units of $x_2$
- Clearly we do not want to use any more than those amounts or else costs will go up for the same level of production
- Then clearly cost function will be

$$
\begin{aligned}
c(w_1, w_2, q) &= w_1 \frac{q}{\alpha} + w_2 \frac{q}{\beta} \\
&= \underbrace{\left( \frac{w_1}{\alpha} + \frac{w_2}{\beta} \right)}_{\text{MC is constant}} q
\end{aligned}
$$

Another way of thinking about this case: we don't care about hammers and friends individually. We only care about a "compound input" comprised of $\alpha$ units of input 1 and $\beta$ units of input 2.

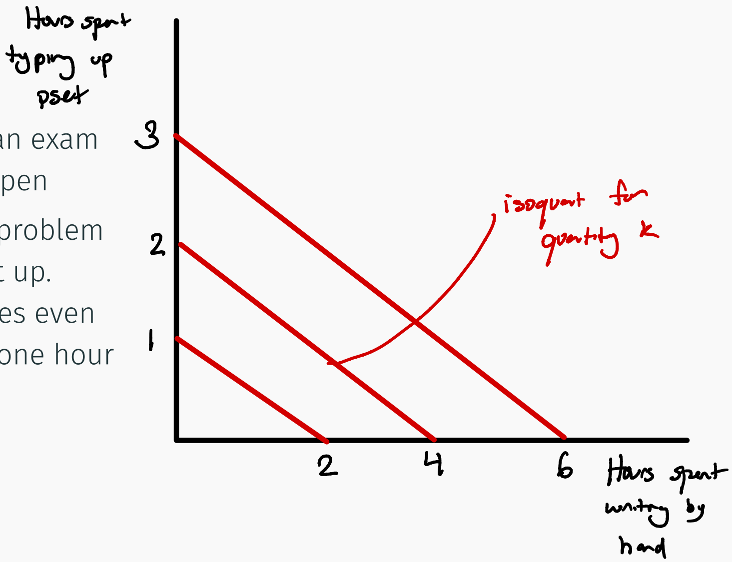
In this case, $\alpha = \beta = 1$ and the compound input is "a friend with their own hammer"

Similar to consumer case: don't care about wheels and handlebars individually. Just bikes.

Here, $\left( \frac{w_1}{\alpha} + \frac{w_2}{\beta} \right)$ is how much it costs a compound input to produce one unit of output.

# Perfect substitutes production: $\alpha x_1 + \beta x_2$

- Example: I can complete an exam using a red pen or a blue pen
- Similar: I can complete a problem set by hand or by typing it up. They are perfect substitutes even though one may take me one hour and the other two hours.

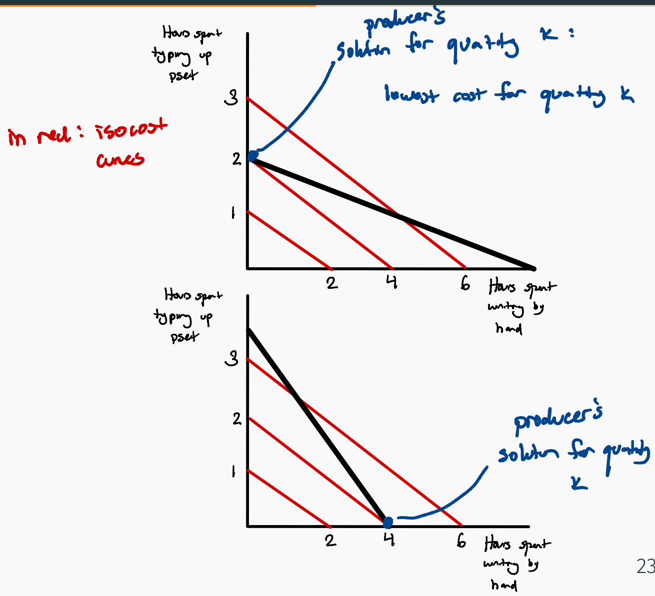


# Perfect substitutes production: $\alpha x_1 + \beta x_2$

- The firm will only use the input that's
  - cost efficient (most productive per unit cost): $\max\left\{\frac{\alpha}{w_1}, \frac{\beta}{w_2}\right\}$
  - or equivalently, least costly per unit product (lowest cost per unit product): $\min\left\{\frac{w_1}{\alpha}, \frac{w_2}{\beta}\right\}$
- Then clearly cost function will be

$$c(w_1, w_2, q) = \min\left\{\frac{w_1}{\alpha}, \frac{w_2}{\beta}\right\} q$$

$$MC(w_1, w_2, q) = \min\left\{\frac{w_1}{\alpha}, \frac{w_2}{\beta}\right\}$$



## Cobb-Douglas production: $f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$

- Unlike in the preference case, magnitudes here matter and have a specific interpretation
- $\alpha$ and $\beta$ measure the relative input intensity of production of the two inputs
    - If $\alpha$ increases relative to $\beta$, production uses more units of input 1 to produce a certain amount
    - These also have interpretations as output elasticities
    - The magnitudes of $\alpha + \beta$ also matter later when we talk about returns to scale

$$y = A x_1^{\alpha} x_2^{\beta}$$

$$\log y = \log A + \alpha \log x_1 + \beta \log x_2$$

$$\frac{\partial \log y}{\partial \log x_1} = \alpha \qquad \frac{\partial \log y}{\partial \log x_2} = \beta$$

Output elasticities of factors of production:

Increasing $x_1$ by 1% leads to $\alpha$ % increase in output

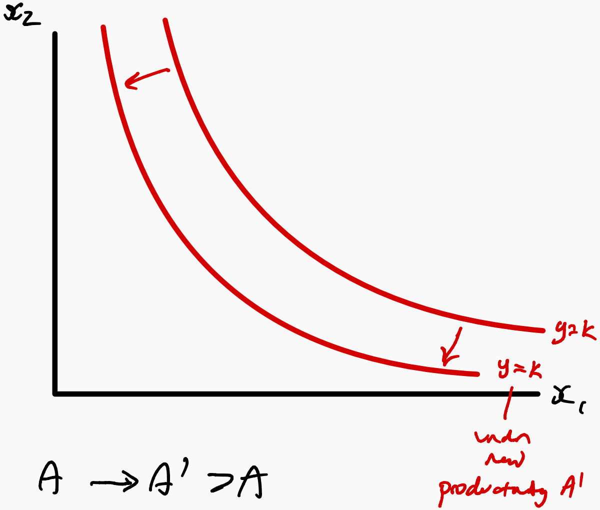
" $x_2$ " $\beta$ %

$\therefore$ Increasing $(x_1, x_2)$ by $100 \times t$ % $\Rightarrow 100 \times (\alpha+\beta)$ % increase in output

The definition of returns to scale! IRS if $\alpha+\beta > 1$; CRS $= 1$; DRS $< 1$

# Cobb-Douglas production: $f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$

- Unlike in the preference case, magnitudes here matter and have a specific interpretation
- $A$ is roughly a measure of the scale of production
    - $\alpha$ and $\beta$ measure the relative production intensity of the two goods while $A$ scales how much of the output good these combine to produce
    - Sort of a multiplier indicating how advanced the technology is
    - We can imagine technology becoming more efficient, which increases $A$



# Example: Cobb-Douglas production: $f(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$

Solving the general case is a bit tedious so skipping steps:

$$x_1^*(w_1, w_2, q) = \left(\frac{A}{q}\right)^{\alpha+\beta} \left(\frac{\alpha w_2}{\beta w_1}\right)^{\frac{\beta}{\alpha+\beta}}$$

$$x_2^*(w_1, w_2, q) = \left(\frac{A}{q}\right)^{\alpha+\beta} \left(\frac{\beta w_1}{\alpha w_2}\right)^{\frac{\alpha}{\alpha+\beta}}$$

# Example: Cobb-Douglas production: $f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$

Multiply by input prices $(w_1, w_2)$ or $(w, r)$ for Labor and Capital:

$$c(w_1, w_2, q) = \left(\frac{A}{q}\right)^{\alpha+\beta} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

$$MC(w_1, w_2, q) = \frac{(\alpha+\beta) A^{\alpha+\beta}}{q^{1-(\alpha+\beta)}} w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

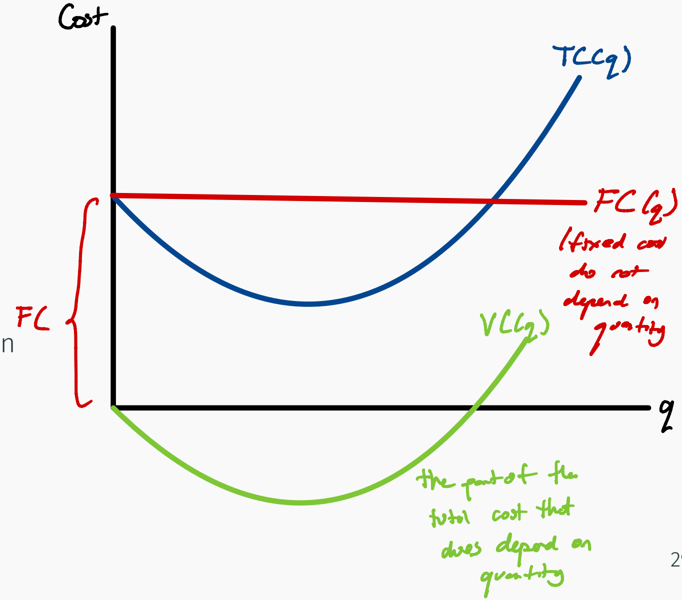
# Other cost functions

# Total costs

The cost function we've derived is called the total cost function, which can be decomposed as such:

$$TC(q) = FC + VC(q)$$

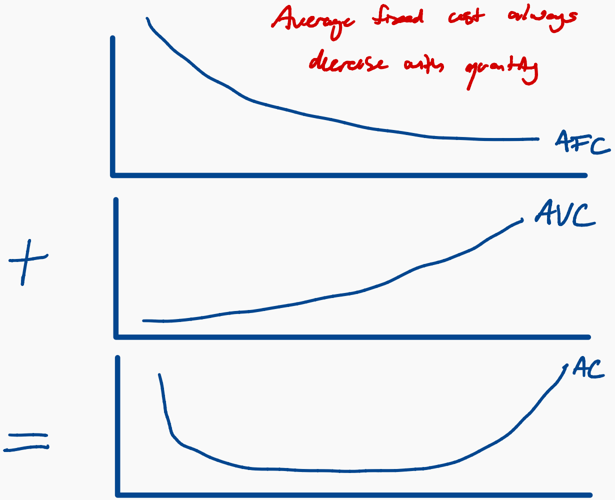
- Fixed costs $FC$
  - The costs that don't depend on quantity
  - "Fixed" in the short run
- Variable costs $VC(q)$
  - The costs that do depend on quantity
  - Can be varied in the short run



# Average costs

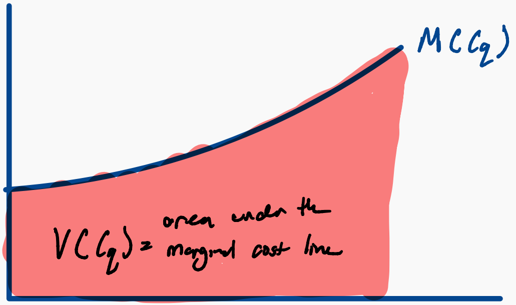
$$AC(q) = \frac{c(q)}{q}$$

- Cost function divided by quantity: the cost per unit of output
- Average cost = average fixed costs + average variable costs



# Marginal costs

$$\begin{aligned} MC(q) &= \frac{dTC(q)}{dq} \\ &= \frac{dFC(q)}{dq} + \frac{dVC(q)}{dq} \\ &= 0 + \frac{dVC(q)}{dq} \\ &= \frac{dVC(q)}{dq} \end{aligned}$$

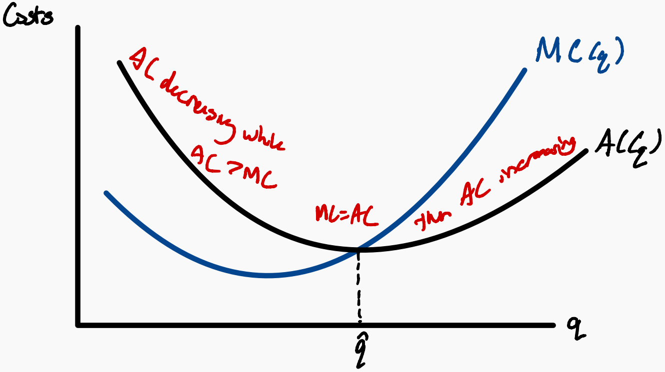



VC(q) = ∫MC(q) dq

- Since fixed costs don't depend on quantity, fixed costs do not affect marginal costs
- Marginal costs are the first derivative of the variable cost

# Relation between average costs and marginal costs

- For $q$ such that $MC(q) < AC(q)$, then average costs are decreasing
- For $q$ such that $MC(q) > AC(q)$, then average costs are increasing
- For $q$ such that $MC(q) = AC(q)$, then average costs are at an inflection point (minimum/maximum)
- Some more in Chapter 22 but cannot fit into this recitation




$$AC'(\hat{q}) = 0$$

$AC''(\hat{q}) > 0$ means local minimum

# Next week: The Producer's Problem II: The Supply Decision

$$\max(\text{Profit}) = \max(\text{Revenue} - \text{Cost}) = \max_q\{p(q)q - c(q)\}$$

Step 2: The supply decision

$$\max_q p(q)q - c(q)$$

Given

- Consumer inverse demand function $q(p)$
- Producer cost function $c(q)$

Derive:

- Marginal revenue $MR(q)$
- Marginal cost $MC(q)$
- Profit-maximizing supply decision $q^* \geq 0$
- Maximal profit $\pi^* = p(q^*)q^* - c(q^*)$