ECON-UN 3211 - Intermediate Microeconomics

Recitation 2: Hicksian demand and special well-behaved preferences

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Quick review: optimization and Marshallian demand

The elements of an optimization problem

$$\max_{\{x_1, x_2\}} / \min \ f(x_1, x_2)$$

$$\text{s.t. } \underbrace{g(x_1, x_2) \ \le k, \ \ge}_{}$$

constraint "binds" if it holds with equality

1. Objective function
2. Choice variables
3. Constraints

Solution: the values of the choice variables that best satisfy the objective function while satisfying the constraints

x_1^* (primitives) $= h_1$ (primitives)

x_2^* (primitives) $= h_2$ (primitives

Asterisk * denotes these are "optimized" in that they solve the optimization problem

The consumer's utility maximization problem

1. Objective function $\max\ u(x_1, x_2) = f(x_1, x_2)$
2. Choice variables $\{x_1, x_2\}$
3. Constraints $\text{s.t. } p_1 x_1 + p_2 x_2 \leq m$

Solution: the affordable bundle that delivers the most utility

- We call this Marshallian demand $x_1^*(p_1, p_2,)$ and $x_2^*(p_1, p_2, m)$
- Equivalently written $x^*(p, m)$ where $x^* = (x_1^*, x_2^*)$ and $p = (p_1, p_2)$ are vectors
- These Marshallian demand functions give us the optimal bundle of a consumer with the given preferences, for any prices $p > 0$ and budget $m > 0$ given

$$x_1^*(p_1, p_2, m) = g_1(p_1, p_2, m)$$
$$x_2^*(p_1, p_2, m) = g_2(p_1, p_2, m)$$

Marshallian demand $x(p, m)$ and indirect utility $v(p, m)$

1. Given prices p and budget m, what is the maximum amount of utility a consumer with the preferences $u(x_1, x_2)$ can achieve?
2. We already have an expression for the quantities that maximize utility for any given p and m: $x^*(p, m)$
3. So just substitute optimal quantities x^* for choice quantities x in the original utility function $u(x_1, x_2)$: $v(p, m) \equiv u(x_1^*(x^*(p, m)))$
4. We distinguish indirect utility using v instead of u to indicate it reflects an optimization result and that it is a function of p and m rather than x_1 and x_2
5. Roy's identity gives us the reverse:

$$x_i^*(p, m) = -\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}$$

Optimization methods for well-behaved preferences: Marshallian demand

1. System of simultaneous equations (see last week's slides for an example)
   1.1 Tangency condition: MRS = price ratio (implicitly comes from assumption of convexity and interior solution)
   1.2 Binding constraint: expenditures = income I (monotonicity)
   1.3 Solve the system of two equations in two unknowns
2. Convert to an unconstrained optimization problem
   2.1 Binding constraint: expenditures = income
      - Solve for one choice variable x_1 in terms of primitives (p_1, p_2, m) and the other choice variable x_2 (or vice versa)
   2.2 Plug this expression for x_1 (or x_2) into the objective function
   2.3 Solve as an unconstrained optimization problem with one first-order condition (tangency condition) then plug into the expression for x_2 (or x_1)
3. The method of Lagrange multipliers (not covered)

The consumer's expenditure minimization problem

1. Objective function $\min \quad p_1 x_1 + p_2 x_2$
2. Choice variables $\{x_1, x_2\}$
3. Constraints s.t. $u(x_1, x_2) = \bar{u}$, some desired level of utility

Solution: the affordable bundle that least expensively achieves utility level $\overline{u}$

- We call this Hicksian demand $x_1^h(p_1, p_2, \overline{u})$ and $x_2^h(p_1, p_2, \overline{u})$
- Use "expenditure" rather than "cost" just because cost in microeconomics is usually associated with production

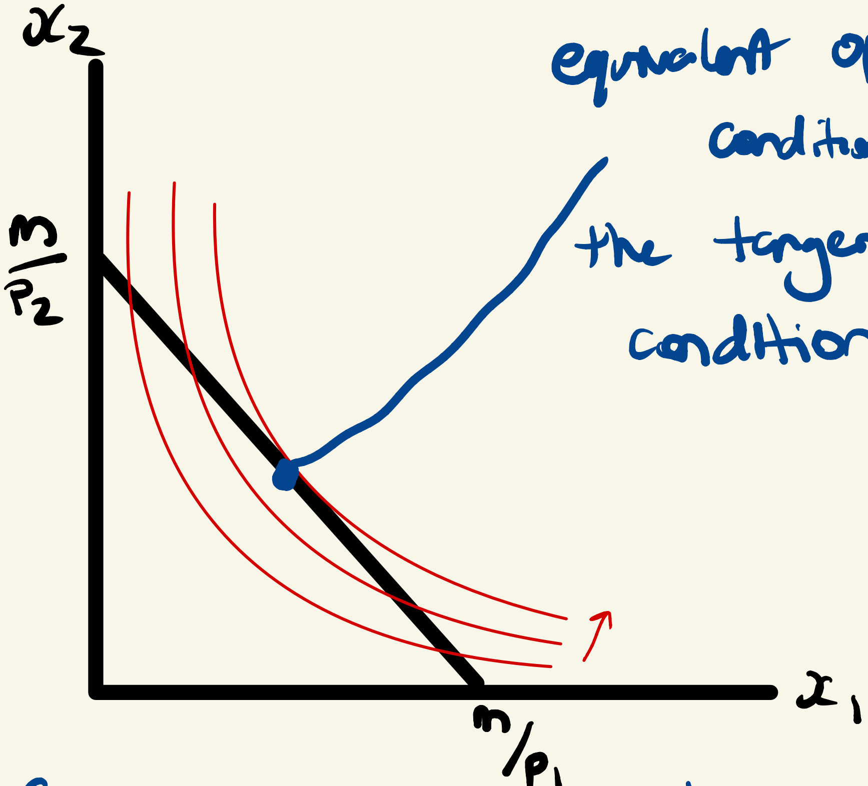
Optimization methods for well-behaved preferences: Hicksian demand

1. System of simultaneous equations
   1.1 Tangency condition: MRS = price ratio **(same tangency condition)**
   1.2 Binding constraint: utility function equals level $\overline{u}$ **(different constraint)**
   1.3 Solve the system of two equations in two unknowns **(primitive $\overline{u}$ instead of m)**
2. Convert to an unconstrained optimization problem
   2.1 Binding constraint: utility function equals level $\overline{u}$ **(different constraint)**
      - Solve for one choice variable x_1 in terms of primitives $(p_1, p_2, \overline{u})$ and the other choice variable x_2 (or vice versa) **(primitive $\overline{u}$ instead of m)**
   2.2 Plug this expression for x_1 (or x_2) into the objective function **(expenditure rather than utility)**
   2.3 Solve as an unconstrained optimization problem with one first-order condition **(same tangency condition)** then plug into the expression for x_2 (or x_1)
3. The method of Lagrange multipliers (not covered)

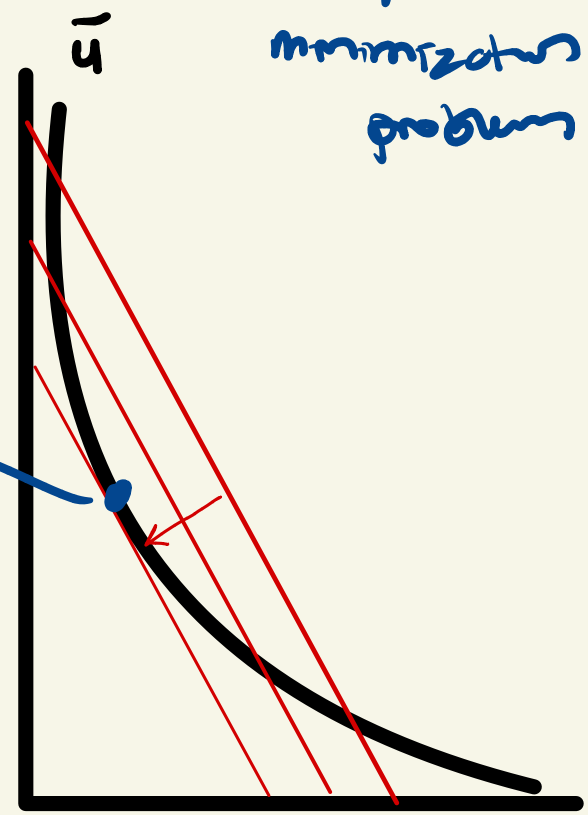
Graphical comparison

The utility maximization problem

The expenditure minimization problem



equivalent optimality condition: the tangency condition



Given budget m, what's the highest indifference curve I can afford to access?

Given desired level of utility $\bar{u}$ which has corresponding indifference curve in black, what's the smallest budget that still lets me access it?

Quick review: preferences and the tangency condition

For "well-behaved" preferences, the tangency condition is *necessary and sufficient* for optimality

$$\text{MRS} = \text{price ratio} \equiv \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \Leftrightarrow \frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

Necessary: the optimal bundles satisfy the tangency condition

1. Preferences have indifference curves everywhere differentiable (no kinks)
2. Preferences have interior solutions only (rules out concave preferences)
3. Necessary, but not sufficient: optimal bundles satisfy the tangency condition but bundles that satisfies the tangency condition may not be optimal

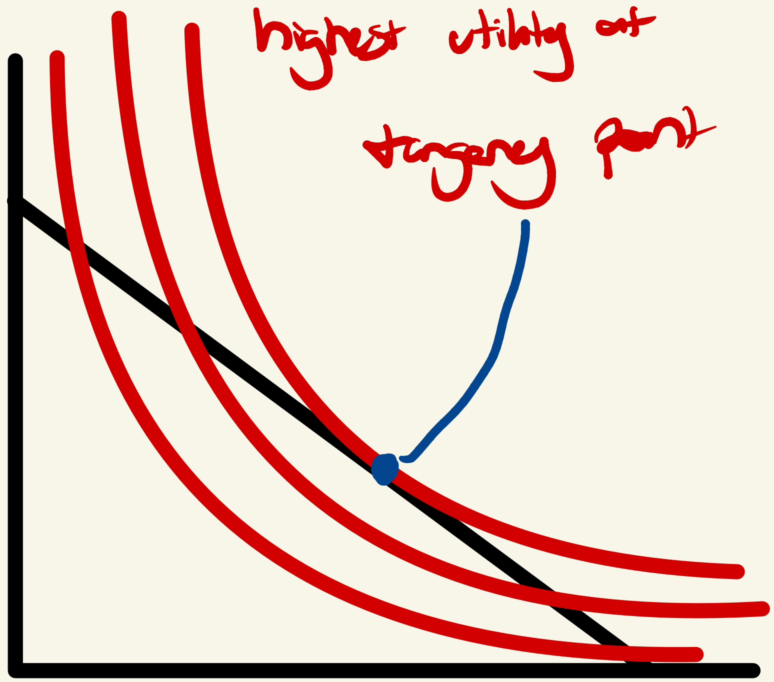
Test your understanding: concave preferences are essentially preferences for bundles of mostly one good: corner solutions

convex preferences like interior solutions (combinations of goods)

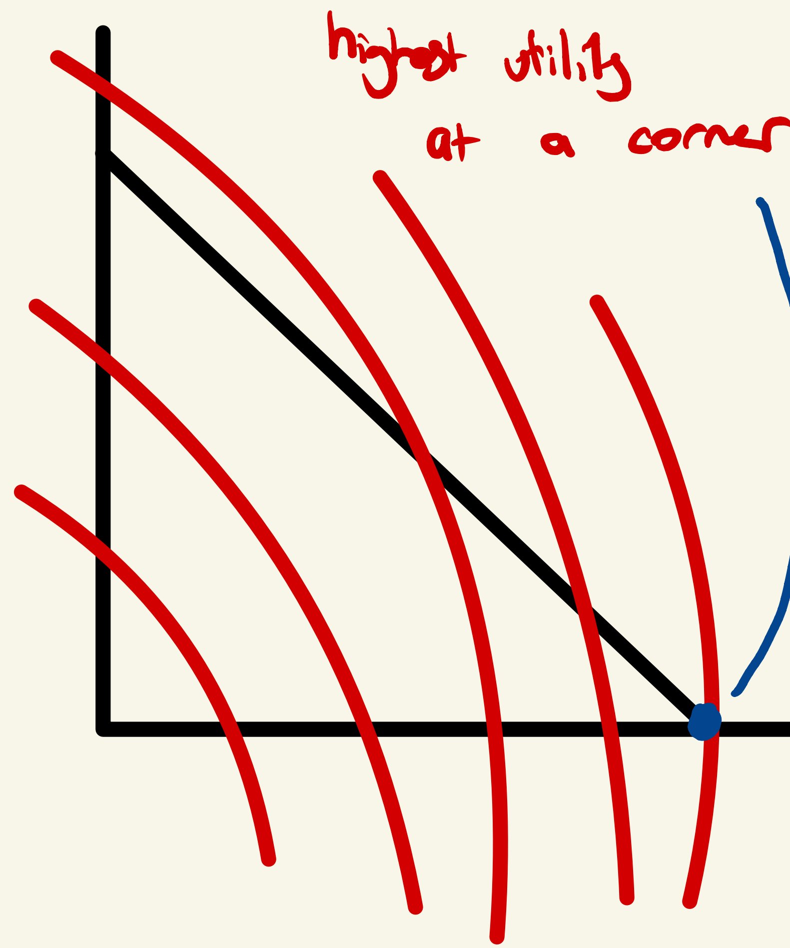
- How does requiring only interior solutions rule out concave preferences?
- Are there well-behaved preferences that are ruled out here?

yes! perfect substitutes, complements

Convex preferences:



Concave preferences:



For "well-behaved" preferences, the tangency condition is *necessary and sufficient* for optimality

$$\text{MRS} = \text{price ratio} \equiv \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \Leftrightarrow \frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

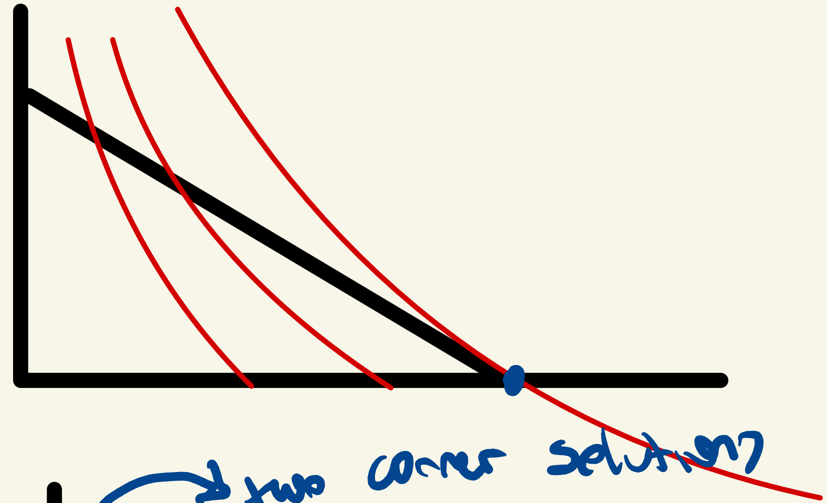
Necessary **and sufficient**: any bundle that satisfies the tangency condition must be optimal

1. Preferences are monotonic
2. Preferences are convex

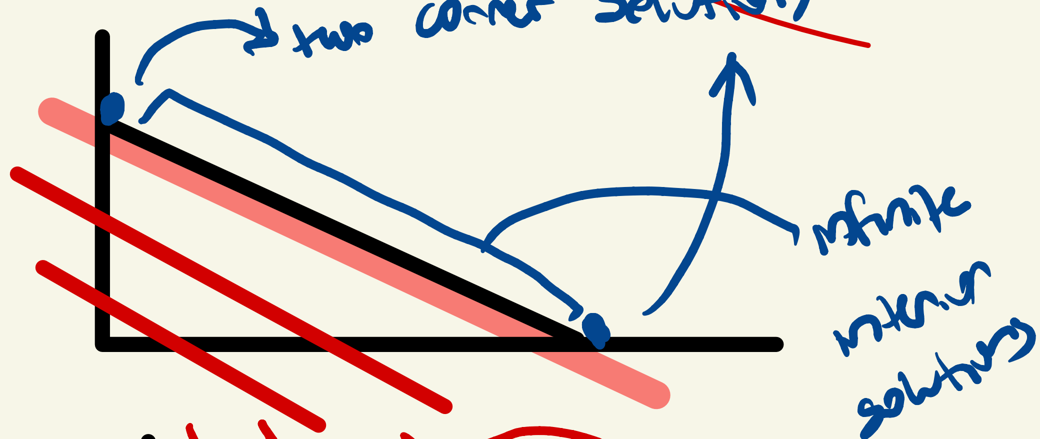
Test your understanding: are there preferences that are…

- convex with exactly one corner solution?
- convex with exactly two corner solutions?
- neither convex nor concave?

One corner solution



Two corner solutions



Neither convex nor concave

Some intervals are strictly convex, some concave

∴ the entire indifference curve is neither

Both convex and concave (weakly)

Linear curves (straight lines)

are both weakly convex and weakly concave

“Well-behaved” preferences guarantee that the tangency condition is *necessary and sufficient* for optimality

$$\text{MRS = price ratio } \equiv \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \Leftrightarrow \frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

Necessary and sufficient **and unique**: if an interior bundle satisfies the tangency condition and it must be the unique optimal choice

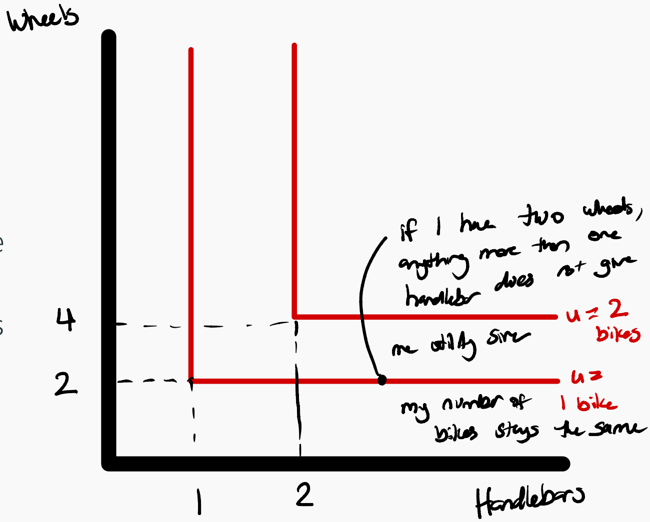
1. Preferences are monotonic
2. Preferences are **strictly** convex

Note: neither non-strict nor strict convexity guarantees the *existence* of an interior bundle that satisfies the tangency condition

Well-behaved preferences beyond the tangency condition

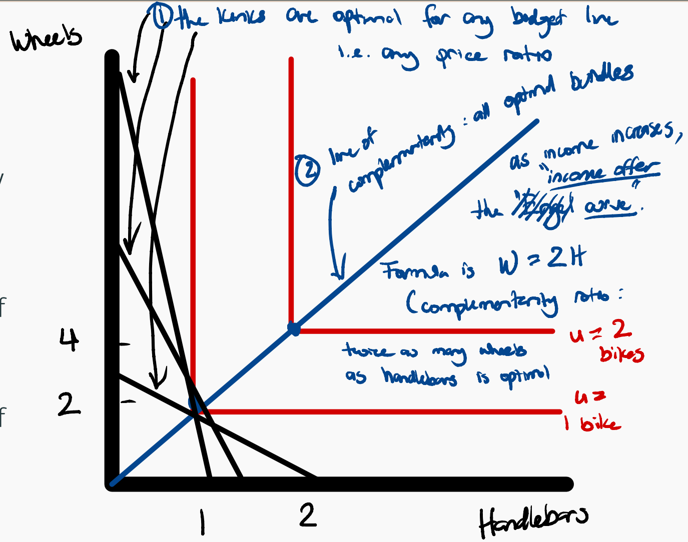
Perfect complements

- Possible examples
  - Left shoes vs. right shoes
  - Rolling paper vs. tobacco
  - Handlebars vs. wheels
- Intuition: we don't really care about the two goods. We only care about the composite good made up of a units of good 2 and b units of good 1
- How do we get from preferences to indifference curves to utility functions?



Intuition: substitution doesn't make sense so the MRS is undefined

- If you're at the kink point, sacrificing any amount will strictly decrease utility
- If you're not at the kink point
  - Sacrificing a marginal amount of the excess good for the other will not change utility
  - Sacrificing a marginal amount of the non-excess good for the other can only decrease utility



Perfect complements: algebraic argument

u (wheel, handlebars) $= u(W, H) = \min\{W, 2H\}$ from the complementarity ratio (CR)

① CR: $W = 2H$, the equation that passes through all kinks / optimal bundles

② BC: $p_W W + p_H H = m$

Substituting ① into ②, $p_W \cdot 2H^* + p_H H^* = m$

$$\Rightarrow H^* = \frac{m}{2p_W + p_H}$$

Plug into ①: $W^* = \dfrac{2m}{2p_W + p_H}$

the optimal bundle

Perfect complements: algebraic argument

For the general perfect complements utility function $u(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$

$\Rightarrow$ Marshallian demand

$$x_1^* = \beta\left(\frac{m}{\beta p_1 + \alpha p_2}\right)$$

$$x_2^* = \alpha\left(\frac{m}{\beta p_1 + \alpha p_2}\right)$$

this means β units of $\underline{x_1}$ for every α units of $\underline{x_2}$

Careful!! NOT α units of x_1 for every β units of x_2 even though intuitive to read the utility function that way

Interpretation: think of a "compound good" where one unit of the "bike" consists of β units of good 1 and α units of good 2.

Its effective "price" is $\beta p_1 + \alpha p_2$ per unit.

Perfect complements: algebraic argument

Thus $\frac{m}{\beta p_1 + \alpha p_2}$ is the "maximum" number

$\Rightarrow$ of "bikes" you can buy with budget m.

α and β determine which share is spent on good 2 vs. good 1

Amount of compound good: $\frac{m}{\beta p_1 + \alpha p_2}$

Amount of good 1 (handlebars) = # of bikes $\times$ # of handlebars per bike

$$= \frac{m}{\beta p_1 + \alpha p_2} \times \alpha$$

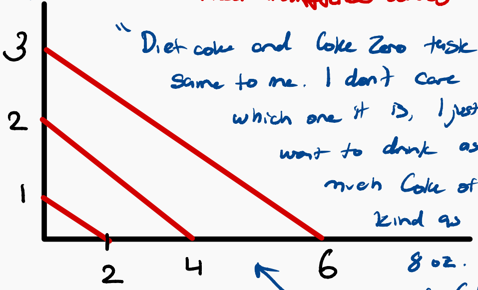
Amount of good 2 (wheels) = # of bikes $\times$ # of wheels per bike

$$= \frac{m}{\beta p_1 + \alpha p_2} \times \beta$$

Perfect substitutes

- Possible examples
  - Organic vs. non-organic bananas
  - Diet Coke vs. Coke Zero
- “I am always indifferent between *a* units of good 2 and *b* units of good 1”
- What does “always” mean here?
- What does this mean for the slope of the indifference curve?

linear indifference curves



“Diet coke and Coke Zero taste the same to me. I dont care which one it is, I just want to drink as much Coke of any kind as possible

linear: slope of indifference curve, i.e., the MRS is a constant

regardless of how much of each good I'm consuming, the rate at which I'd prefer one to the other is the same, a constant.

Perfect substitutes

Just remember this form

The utility function must be linear in both goods

$u(x_1, x_2) = \alpha x_1 + \beta x_2 + c$

$$MRS(x_1, x_2) = \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{MU_1(x_1, x_2)}{MU_2(x_1, x_2)} = k$$

- If the MRS is constant, then what form does the utility function have?

Just extra, don't need to learn this!

Why? Suppose not. Then at least one of MU_1 and MU_2 is a function of x_1 and/or x_2

① If one is a constant and the other a function $f(x_1, x_2)$ then their ratio cannot be a constant.

② If both are functions of x_1 and/or x_2, then their ratio can only be constant if they are linear functions of one another:

$MU_1(x_1, x_2) = k MU_2(x_1, x_2)$

and $\int MU_1(x_1, x_2)\, dx_1 = \int MU_2(x_1, x_2)\, dx_2$

$= \int \frac{1}{k} MU_1(x_1, x_2)\, dx_2$

This is only possible if goods 1 and 2 are the same good.

Perfect substitutes

Recall Pset 2:

if MRS (x_1, x_2) > price ratio, then optimal to reallocate budget to good 1

$$\frac{MU_1}{MU_2} \text{ vs. } \frac{p_1}{p_2}$$

< good 2

$$\Leftrightarrow \frac{MU_1}{p_1} \text{ vs. } \frac{MU_2}{p_2}$$

= , then budget allocation is optimal (for interior solutions)

With perfect substitutes, all four numbers here are constant!

So if LHS > RHS, true for all values of (x_1, x_2)
<
=

- If LHS > RHS, then it is utility-increasing to spend more on good 1
- If LHS < RHS, spend more on good 2

corner solutions we associate with concavity

- If LHS = RHS, then the consumer is indifferent

the same two corner solutions PLUS (infinite) interior solutions we associate w/ convexity

Perfect substitutes preferences are both concave and convex!

Perfect substitutes

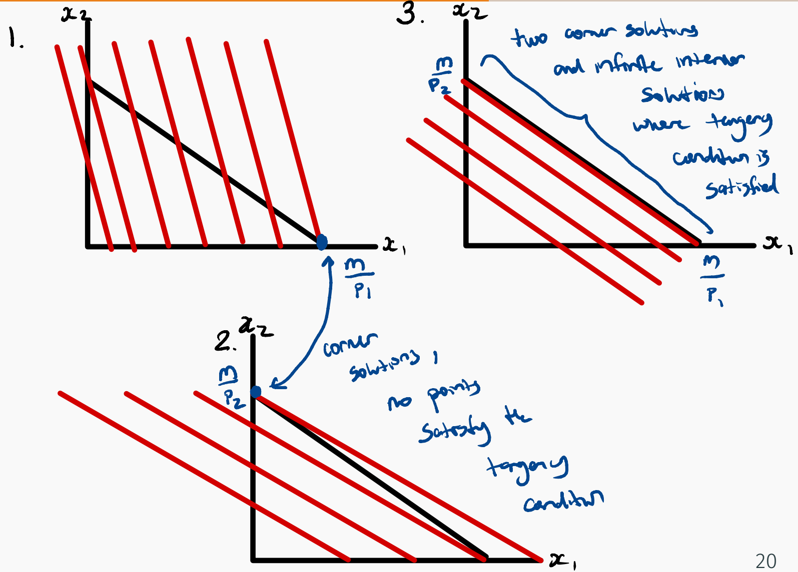
Recall Pset 2:

$$\frac{MU_1}{MU_2} \text{ vs. } \frac{p_1}{p_2}$$

$$\Leftrightarrow \frac{MU_1}{p_1} \text{ vs. } \frac{MU_2}{p_2}$$

The three cases visualized

1. MRS $\equiv \frac{\alpha}{\beta}$ > price ratio
2. MRS $\equiv \frac{\alpha}{\beta}$ < price ratio
3. MRS $\equiv \frac{\alpha}{\beta}$ < price ratio



Perfect substitutes and the demand function

Optimal bundle depends on relative prices

- Prices p_1, p_2 and budget m
- Preferences: $u(x_1, x_2) = \alpha x_1 + \beta x_2 + c$
- Then what does the demand function depend on?

Three cases so we write our Marshallian demand as a <u>piecewise function</u>

$$x_1^*(p,m) = \begin{cases} \frac{m}{p_1} & \text{if } \frac{\alpha}{\beta} > \frac{p_1}{p_2} \\ \text{any value } \left[0, \frac{m}{p_1}\right] & \text{if } \frac{\alpha}{\beta} = \frac{p_1}{p_2} \\ 0 & \text{if } \frac{\alpha}{\beta} < \frac{p_1}{p_2} \end{cases}$$

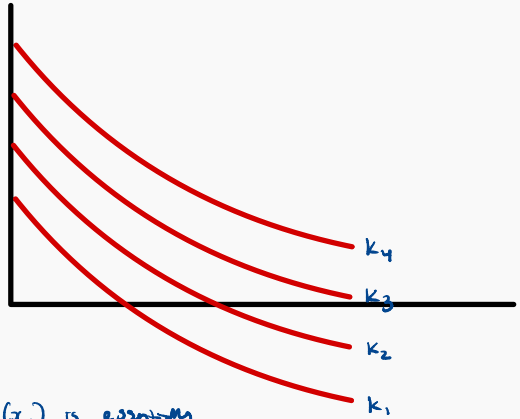
$$x_2^*(p,m) = \begin{cases} 0 & \text{if } \frac{\alpha}{\beta} > \frac{p_1}{p_2} \\ \text{any value } \left[0, \frac{m}{p_2}\right] & \text{if } \frac{\alpha}{\beta} = \frac{p_1}{p_2} \\ \frac{m}{p_2} & \text{if } \frac{\alpha}{\beta} < \frac{p_1}{p_2} \end{cases}$$

Quasilinear preferences

- Suppose indifference curves are just vertical translations of one another
- It then follows that indifference curves have the form $x_2 = k - v(x_1)$
  - k is a constant unique to each level of utility
  - some function v of x_1 giving the shape displayed here
- Intuitive to index utility by k ⇒ $x_2 - v(x_1)$ is essentially the utility function measured as k
- "Quasilinear": linear in one good, maybe not the other

$u(x_1, x_2) = x_2 - v(x_1)$

if $v(x_1)$ is linear, we get a special case of quasilinear preferences → perfect substitutes!



Quasilinear preferences

$$\max_{\{x_1,x_2\}} v(x_1) + x_2 \quad \text{(OF)}$$

$$\text{s.t.} \quad p_1x_1 + p_2x_2 \leq m \quad \text{(BC)}$$

From BC : $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$

Sub into OF and transform into unconstrained optimization problem

$$\max_{\{x_1\}} = v(x_1) + \frac{m}{p_2} - \frac{p_1}{p_2}x_1$$

Take first-order condition wrt x_1 :

$$FOC_{x_1} : 0 = v'(x_1) + 0 - \frac{p_1}{p_2}$$

$$\Rightarrow v'(x_1) = \frac{p_1}{p_2}$$

Note that income m does not affect the solution!

For a given form of the utility function, this will allow you to derive the Marshallian demand functions

Quasilinear preferences: ~~the~~ “zero income effect”

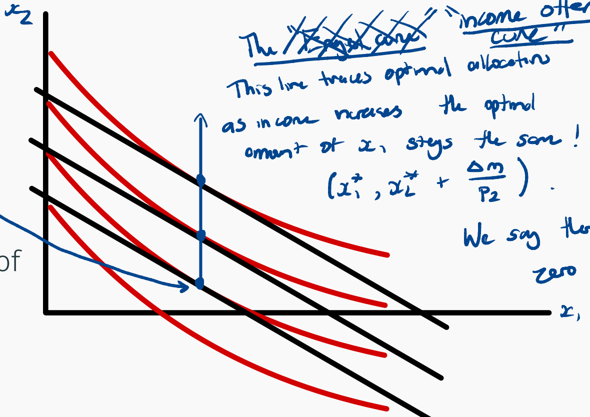
Inverse demand function — from rearranging the FOC in previous slide

$$p_1(x_1) = p_2 v'(x_1)$$

price as a function of quantity

Since m drops out, the bundle that satisfies the tangency condition is independent of m.

- Suppose the tangency condition is met at some interior bundle (x_1^*, x_2^*)
- What happens if we shift the budget line?
- What if we were to trace the optimal bundle as a function of income?

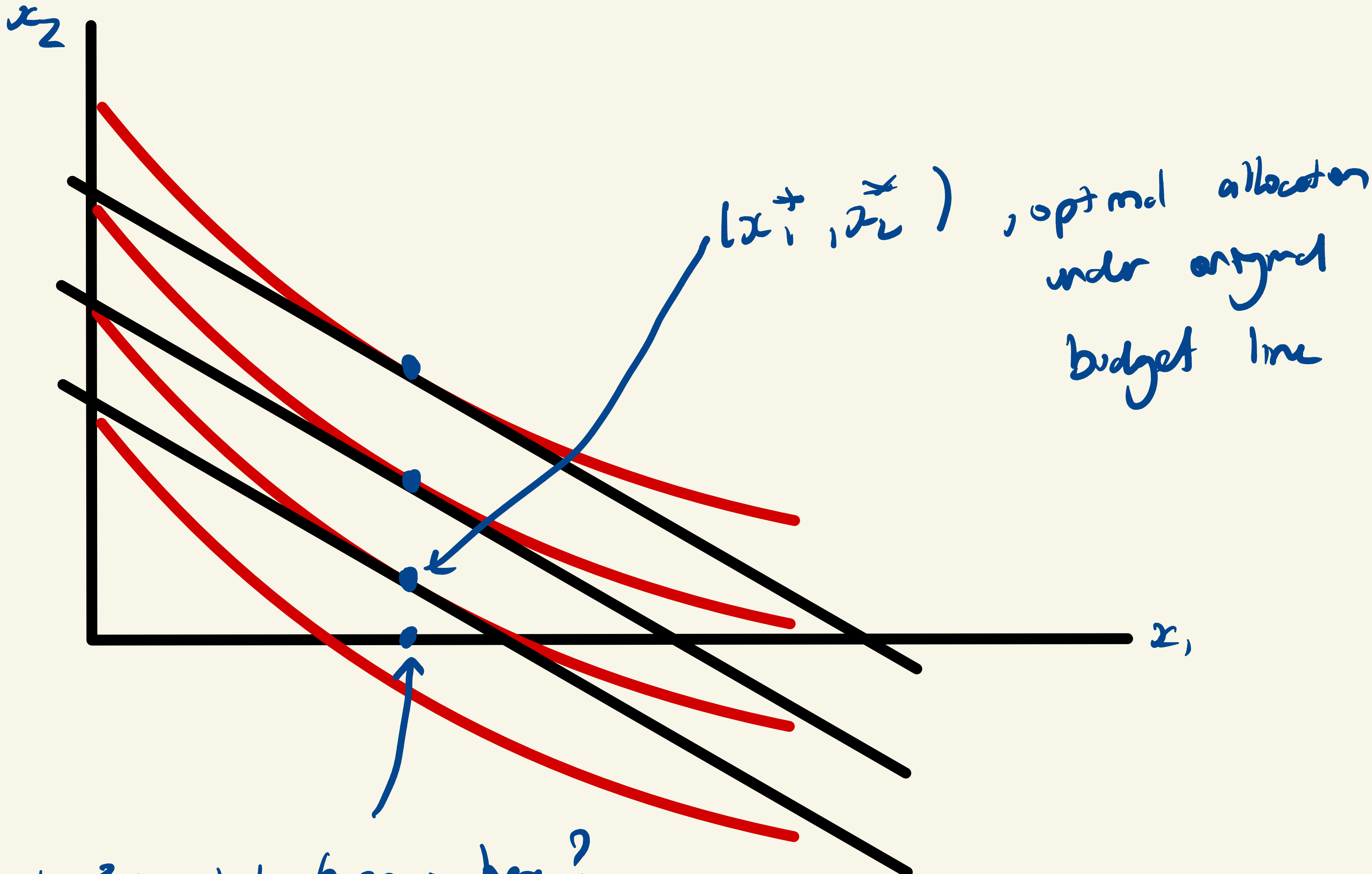


The ~~tangent curve~~ “income offer curve”

This line traces optimal allocations as income increases the optimal amount of x_1 stays the same!

$$\left(x_1^*, x_2^* + \frac{\Delta m}{p_2}\right).$$

We say there is zero income effect



Pset 3: what happens here?

our solutions $\left(x_1^*, x_1^* + \frac{\Delta m}{p_2}\right)$ only give us the bundle that satisfies the tangency condition.

What about when we hit corners?

HINT: Think about the piecewise solutions for perfect substitutes!

Quasilinear preferences: ~~Th~~e “zero income effect”

- How can optimal demand for good 1 be independent of income if at income 0, we can’t afford any of either good and optimal demand must be zero?
- An example: starting at zero income, there is a range of subsistence incomes where you spend any earnings you have on essentials like toothpaste
- But after a certain point, increases in income aren’t likely to make you use more toothpaste and you might spend the “excess” earnings on something like vacation or recreation
- Pset 3: clearly we have to consider multiple cases corresponding to interior and corner solutions. What is the relevant threshold income level at which point the zero income effect kicks in?

Bonus question: What does the complete income effect curve and Engel Curve look like?

Time permitting or problems to work through on your own

Given preferences $u(x) = \min\{\alpha x_1, \beta x_2\}$

1. Solve for Marshallian demands
2. Solve for indirect utility
3. Verify Roy's identity for good 1
4. Solve for Hicksian demands and the expenditure function

Given preferences $u(x) = \min\{\alpha x_1, \beta x_2\}$

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