ECON-UN 3211 - Intermediate Microeconomics

Recitation 1: The consumer's optimization problem
Budget sets, utility functions, and the optimality condition

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The budget line

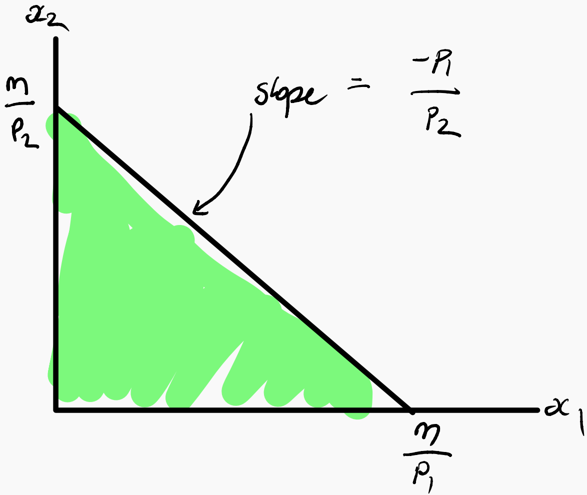
The budget line

- Income m
- Quantities x_1 and x_2
- Prices p_1 and p_2
- Budget set

$$x_1 p_1 + x_2 p_2 \leq m$$

- Budget line

$$\begin{aligned} m &= x_1 p_1 + x_2 p_2 \\ \Rightarrow x_2 &= \frac{m}{p_2} - \frac{p_1}{p_2} x_1 \end{aligned}$$



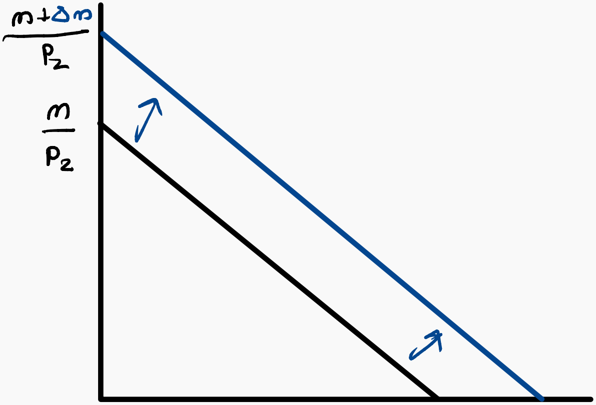
The budget line: changes in income

- Income $m \rightarrow m + \Delta m$
- Quantities x_1 and x_2
- Prices p_1 and p_2
- Budget set

$$x_1 p_1 + x_2 p_2 \leq m + \Delta m$$

- Budget line

$$\begin{aligned} m + \Delta m &= x_1 p_1 + x_2 p_2 \\ \Rightarrow x_2 &= \frac{m + \Delta m}{p_2} - \frac{p_1}{p_2} x_1 \end{aligned}$$



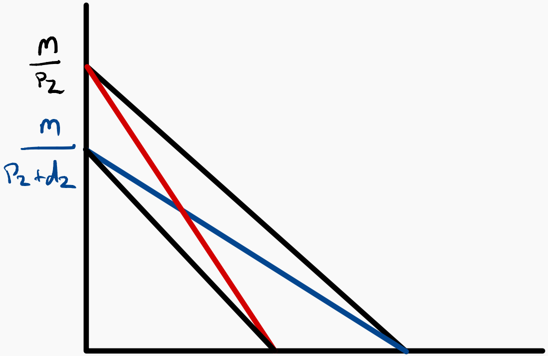
The budget line: changes in prices

- Income m
- Quantities x_1 and x_2
- Prices $p_1 \to p_1 + d_1$ and $p_2 \to p_2 + d_2$
- Budget set

$$x_1(p_1 + d_1) + x_2(p_2 + d_2) \leq m$$

- Budget line

$$\begin{aligned} m &= x_1(p_1 + d_1) + x_2(p_2 + d_2) \\ \Rightarrow x_2 &= \frac{m}{p_2 + d_2} - \frac{p_1 + d_1}{p_2 + d_2} x_1 \end{aligned}$$



Representing preferences

Preferences and utility representations

- Utility function
  - Input: quantity x or bundle $(x_1, x_2, \ldots)$
  - Output: a number
- A utility function represents preferences if $u(x) > u(y)$ whenever x is preferred to y
- Utility representations of preferences are ordinal not cardinal: how much larger $u(x)$ is than $u(y)$ is meaningless

Consumer prefers A to B and B to C

$\Rightarrow$ A to C

$v(x) = 2u(x)$, $w(x) = v(x) + 3000$

Bundle	$u(x)$	$v(x)$	$w(x)$
C	5	10	3010
B	10	20	3020
A	300	600	3600

Preferences are invariant to (positive) monotonic transformations of utility

- If $u(x)$ is a valid representation, then so is $v(x)$ if $v(x) = f(u(x))$ for a monotonic function f
    - Adding/subtracting
    - Multiplying/dividing
    - Exponentiating
    - Taking logs

Marginal utility and marginal rate of substitution

- Marginal utility: the additional utility derived from consuming one additional unit
- Marginal rate of substitution: rate at which a consumer can give up some amount of good 2 in exchange for good 1 while maintaining the same level of utility (slope of the indifference curve)

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} \qquad MU_2 = \frac{\partial u(x_1, x_2)}{\partial x_2}$$

(x_1, x_2) Bundle 1 $\qquad (x_1 + dx_1, x_2 + dx_2)$ Bundle 2

$$du = MU_1 \times dx_1 + MU_2 \times dx_2 = 0$$

$$\Rightarrow MU_1 dx_1 = -MU_2 dx_2$$

$$\Rightarrow \frac{dx_1}{dx_2} = -\frac{MU_1}{MU_2}$$

MRS is invariant to to (positive) monotonic transformations of utility

- Suppose we have $v(x_1, x_2) = f(u(x_1, x_2))$ for monotonic function f
- If MRS is the same, then the shape of the indifference curves are the same
- Thus preferences have infinite utility function representations but uniquely shaped indifference curves

Example: Cobb-Douglas utility functions

$MU_1 = \frac{c}{x_1} \quad MU_2 = \frac{d}{x_2}$

$$MRS_u = -\frac{MU_1}{MU_2} = \frac{-(c/x_1)}{(d/x_2)} = -\frac{c}{d} \times \frac{x_2}{x_1}$$

$$\begin{aligned}
u(x_1, x_2) &= c \log x_1 + d \log x_2 \\
v(x_1, x_2) &= \exp(u(x_1, x_2)) \\
&= \exp(c \log x_1 + d \log x_2) \\
&= \exp(\log x_1^c + \log x_2^d) \\
&= x_1^c x_2^d
\end{aligned}$$

$MV_1 = c x_1^{c-1} x_2^d$

$MV_2 = d x_1^c x_2^{d-1}$

$$MRS_v = \frac{-MV_1}{MV_2} = -\frac{(c x_1^{c-1})(x_2^d)}{(d x_1^c)(x_2^{d-1})}$$

$$= -\frac{c}{d} \left[x_1^{-1}\right]\left[x_2^{1}\right]$$

$$= -\frac{c}{d} \times \frac{x_2}{x_1}$$

Elements

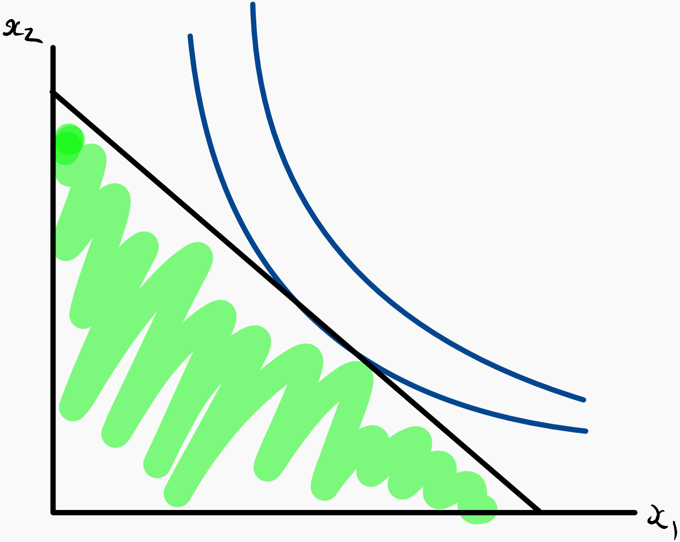
The elements of an optimization problem

In general:

1. Objective function
2. Choice variables
3. Constraint

$$\max_{\{x_1, x_2\}} u(x_1, x_2)$$

$$\text{s.t. } x_1 p_1 + x_2 p_2 \leq m$$



The tangency condition

The tangency condition: greatest indifference curve tangent to the budget line

indifference curve that's tangent to the budget line

$$MRS = \frac{p_1}{p_2}$$

The tangency condition

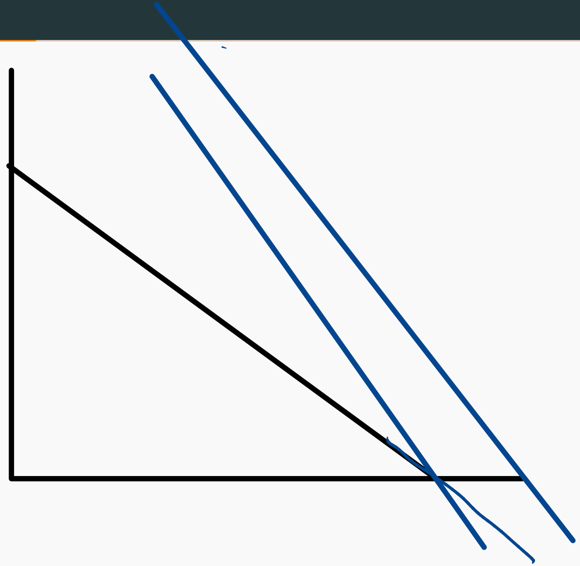
Exceptions:

- Intersection at a kink
- Intersection at bundles at ≤ 0 quantities

Ruling these out:

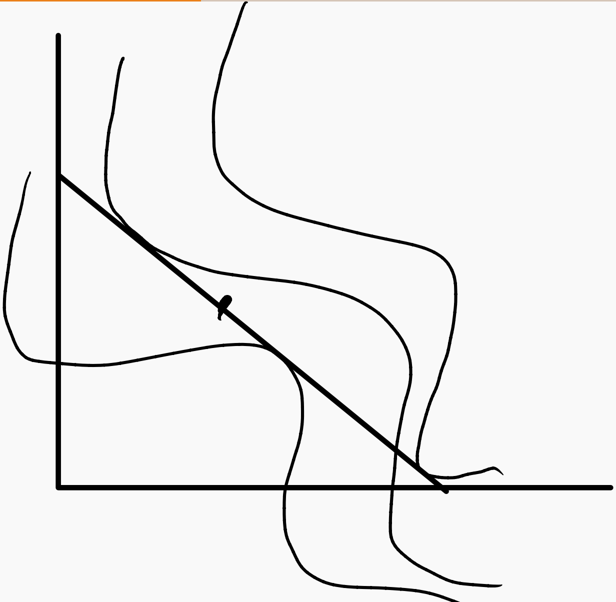
- Smooth indifference curves (differentiable utility functions)
- Restrict to interior optima

Then the tangency condition is **NECESSARY** for optimality



The tangency condition

What about concave preferences?



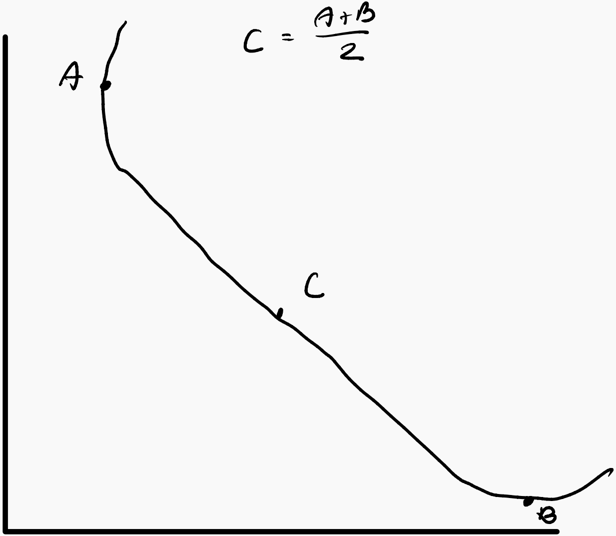
The tangency condition

Assumptions for “well-behaved” preferences

1. Monotonocity: more of any good gives more utility
2. Convexity

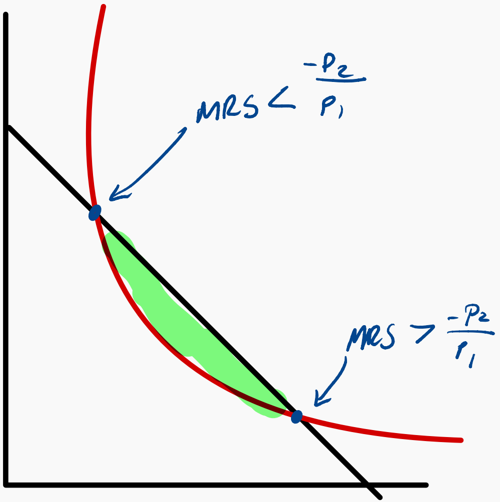
Then the tangency condition is **NECESSARY AND SUFFICIENT** for optimality

- But: not necessarily unique: for that, need *strict* convexity



The tangency condition: in economic terms

- Slope of the indifference curve = slope of the budget line (i.e., the price ratio)
    - MRS: the amount of additional good 2 that would deliver the same utility as another unit of good 1
    - Slope of budget line: the rate at which the market relatively values good 2 to good 1
- What if these were not equal?



Deriving the demand functions

Deriving a demand function: intuition

- (Exogenously) given economic environment
  1. Prices p_1 and p_2
  2. Budget m
  3. Preferences $u(x_1, x_2)$
- We want functions $x_1^*(p_1, p_2, m)$ and $x_2^*(p_1, p_2, m)$ that give us the optimal bundle $(x_1^*(p_1, p_2, m), x_2^*(p_1, p_2, m))$

Deriving a demand function: intuition

- To do this, we compute “first-order” conditions for optimality
  - “First-order” because they entail first derivatives: of the indifference curve (MRS) and of the budget line (price ratio)
  - Condition 1: the budget constraint must bind
  - Condition 2: the slope of the indifference curve equals the slope of the budget line

$$
\begin{aligned}
\text{MRS} &= \text{Slope of budget line (times negative one)} \\
\Leftrightarrow \frac{MU_1}{MU_2} &= \text{Price ratio} \\
\Leftrightarrow \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} &= \frac{p_1}{p_2}
\end{aligned}
$$

- Two conditions give us two equations in two unknowns x_1 and x_2: the solution is a unique bundle unless they are linearly dependent

Deriving a demand function: intuition

$$\max_{\{x_1,x_2\}} u(x_1, x_2)$$

$$\text{s.t.} \quad x_1 p_1 + x_2 p_2 \leq m$$

Two main ways of solving this constrained optimization problem

1. Convert to an unconstrained optimization problem
    - Take the budget line equation and express x_1 in terms of x_2 (or vice versa)
    - Substitute into the objective function
    - Solve like an optimization problem with one choice variable and no constraint

$$\max_{\{x_1\}} u(x_1)$$

    - Back out the other quantity by substituting the resultant x_1 into the budget constraint
2. The method of Lagrange multipliers (not covered in our treatment)

Example: Cobb-Douglas preferences

$c \log x_1 + d \log x_2$

Preferences:

$$u(x_1, x_2) = x_1^c x_2^d$$

So the problem is:

$$\max_{\{x_1, x_2\}} x_1^c x_2^d$$

s.t. $x_1 p_1 + x_2 p_2 \leq m$

$\Rightarrow x_1 p_1 + x_2 p_2 = m$

$\Rightarrow x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$

$$\max_{x_1, x_2} x_1^c x_2^d$$

[we can take logs of the objective function to make the math easier]

$$= \max_{x_1, x_2} c \log x_1 + d \log x_2$$

substitute x_2

$$= \max_{x_1} c \log x_1 + d \log \left[\frac{m}{p_2} - \frac{p_1}{p_2} x_1 \right]$$

$$* \text{FOC}_{x_1}: \frac{c}{x_1} + \frac{d \left[-\frac{p_1}{p_2} \right]}{\frac{m}{p_2} - \frac{p_1}{p_2} x} = 0$$

[apply log derivative rule — see notes]

$$\Rightarrow \frac{c}{x_1} = \frac{d p_1}{m - p_1 x_1} \Rightarrow cm - c p_1 x_1 = d p_1 x_1$$

$$\Rightarrow x_1 = \frac{m}{p_1} \left[\frac{c}{c+d} \right]$$

Example: Cobb-Douglas preferences

Third method : system of two equations

Preferences:

$$u(x_1, x_2) = x_1^c x_2^d$$

So the problem is:

$$\max_{\{x_1, x_2\}} x_1^c x_2^d$$

$$\text{s.t.} \quad x_1 p_1 + x_2 p_2 \leq m$$

① Budget constraint binds : $m = x_1 p_1 + x_2 p_2$

② Tangency condition : $\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$

Let $v = \log(u(x_1, x_2))$ for convenience : $v(x_1, x_2) = c \log x_1 + d \log x_2$

$$\frac{MV_1}{MV_2} = \frac{c/x_1}{d/x_2} = \frac{p_1}{p_2} \iff x_2 = \frac{p_1}{p_2} \frac{d x_1}{c}$$

Solve as simultaneous equations : plug ② into ①

$$m = x_1 p_1 + p_2 \left[\frac{p_1}{p_2} \frac{d x_1}{c} \right]$$

$$= x_1 p_1 + p_1 \frac{d x_1}{c} = x_1 p_1 \left(1 + \frac{d}{c}\right)$$

$$\Rightarrow x_1 = \frac{m}{p_1 \left(1 + \frac{d}{c}\right)} = \frac{mc}{p_1 (c+d)} = \frac{m}{p_1} \cdot \frac{c}{c+d}$$

Example: Cobb-Douglas preferences

Preferences:

$$u(x_1, x_2) = x_1^c x_2^d$$

So the problem is:

$$\max_{\{x_1, x_2\}} x_1^c x_2^d$$

s.t. $x_1 p_1 + x_2 p_2 \leq m$

In either method, can then solve for x_2 to get the same solution

$$x_1^* = \left(\frac{m}{p_1}\right) \frac{c}{c+d} \quad \text{and} \quad x_2^* = \left(\frac{m}{p_2}\right) \cdot \frac{d}{c+d}$$

$\left(\frac{m}{p_1}\right)$: amount of good 1 if you spent your entire income only on good 1

$\frac{c}{c+d}$, $\frac{d}{c+d}$: relative size of the exponents in the utility function

$\left(\frac{m}{p_2}\right)$: amount of good 2 if you spent your entire income only on good 2

Key points for Cobb-Douglas:

- the relative size of c vs. d determines how much of your budget you'll spend on good 1 vs. good 2
- the prices p_1 and p_2 do not affect the share of your budget going to good 1! Only affects how many units you can buy with that share!