ECON-UN 3211 - Intermediate Micro Recitation 1

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1 Basics and notation

- A function f is a rule or mapping that takes in input, say x, and produces a unique output $f(x)$. This input can be single-dimensional or multi-dimensional, say $x_1, x_2, ..., x_n$; in the latter case, we can say we are evaluating the function f for n variables evaluated at values $x_1, x_2, ..., x_n$ to obtain the output $f(x_1, x_2, ..., x_n)$.
- Throughout this course, we'll be making use of functions to describe relationships of the following variety:

$$y = f(x_1, x_2, ..., x_n) \tag{1}$$

  In this case, we can say, "the dependent variable y is a function of the independent variables $x_1, x_2, ..., x_n$." This terminology captures the direction of dependence between the variables: the output is determined by the input. In the $n = 2$ case, y is determined by x_1 and x_2. This is what we mean when we say "y is a function of x_1 and x_2".
- Types of functions (informal definitions)
  - Continuous: a function that can be drawn without lifting a pencil from the paper
  - Smooth: a function that has no kinks or corners. Slightly more formally, a function whose derivatives are continuous and without holes.
  - Monotonic: a function that is always increasing or always decreasing with respect to its inputs. A function can be monotonically increasing with respect to one input and monotonically decreasing with respect to another. For example, the price of a second-hand car may be monotonically increasing with respect to its size but monotonically decreasing with respect to its age.
  - Inverse: if f is a function of input x, g is the inverse function of f if $g(f(x)) = x$. For example, if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$ and so we say g is the inverse of f when $x \geq 0$ because g "undoes" what f does to x.
- As an example, we might think of y as the quantity of cars produced, L as the quantity of labor used in production, and K as the quantity of capital used in production. This two-dimensional function of the independent variables L and K thus describes a very basic production function where the number of cars a firm produces is determined by the quantity of the two inputs used.
- The above function f is unspecified; we have not given it a specific form. A simple production function specification used in economics is the two-input Cobb-Douglas production function with constant returns to scale:

$$y = AL^{\alpha}K^{1-\alpha} \tag{2}$$

  for fixed constants $A > 0$ and $0 < \alpha < 1$, whose interpretations we'll investigate later int his course.
- Returning to the first bullet point, it's important to emphasize the word "unique" in our definition. A bundle of inputs $(x_1, ..., x_n)$ delivers only one—i.e., "unique"—value of $f(x_1, ..., x_n)$. In the car production example, this would be like saying using 5 units of L and 6 units of K as inputs in production enables the production of $f(5, 6)$ cars, no more and no less.
- What is *not* necessarily unique is the set of inputs which can deliver the same output. In fact, microeconomics work is very interested in the set of inputs that deliver the same output: if f were a utility function whose inputs are a bundle of goods and whose output is the amount of utility it grants a consumer, then plotting the set of bundles that deliver the exact same utility traces out what we call an indifference curve. Likewise, if f were an expenditure function whose inputs are a bundle of goods the consumer can spend on and whose output is the amount spent on the given bundle, then plotting the set of bundles that exhausts their budget exactly will trace out what we call a budget constraint.

2 Derivatives of common functions

$$\begin{aligned}
\frac{dc}{dx} &= 0 \\
\frac{dx}{dx} &= 1 \\
\frac{dx^2}{dx} &= 2x \\
\frac{d}{dx}e^x &= e^x \\
\frac{d}{dx}\ln(x) &= \frac{1}{x} \\
\frac{d}{dx}\log_a(x) &= \frac{1}{x\ln(a)}
\end{aligned} \tag{3}$$

3 Derivative rules

- **Power rule and polynomials**: $\frac{dax^n}{dx} = nax^{n-1}$

  Example:

$$\frac{d}{dx}(4x^3 + 3x^2 + 8x + 5) = 12x^2 + 6x + 8 \tag{4}$$

- Product rule: $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$

Example:

$$
\begin{aligned}
f(x) &= x \\
\Rightarrow f'(x) &= 1 \\
g(x) &= \ln(x) \\
\Rightarrow g'(x) &= \frac{1}{x} \\
\frac{d}{dx}[f(x)g(x)] &= \frac{d}{dx} x \ln(x) \\
&= x\frac{1}{x} + 1 \times \ln(x) \\
&= 1 + \ln(x)
\end{aligned} \tag{5}
$$

- **Quotient rule**: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$

  Example:

$$
\begin{aligned}
f(x) &= x \\
\Rightarrow f'(x) &= 1 \\
g(x) &= \ln(x) \\
\Rightarrow g'(x) &= \frac{1}{x} \\
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(\frac{x}{\ln(x)}\right) \\
&= \frac{1 \times \ln(x) - \frac{1}{x} \times x}{\ln(x)^2} \\
&= \frac{1}{\ln(x)} - \frac{1}{\ln(x)^2}
\end{aligned} \tag{6}
$$

- **Chain rule**: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \times g'(x)$

  Example:

$$
\begin{aligned}
f(x) &= x^8 \\
\Rightarrow f'(x) &= 8x^6 \\
g(x) &= 3x + 2 \\
\Rightarrow g'(x) &= 3 \\
\frac{d}{dx}[f(g(x))] &= f'(g(x)) \times g'(x) \\
&= 8(3x+2)^7 \times 3 \\
&= 24(3x+2)^7
\end{aligned} \tag{7}
$$

- Higher-order derivatives
  - Second derivatives are straightforward: simply take the derivative of the original function and then take that derivative's derivative. If the first derivative is the rate of change of a function with respect to an input, the second derivative is thre ate of change of the rate of change of the function with respect to that input.
  - For example, if t is time and $f(t)$ gives the distance traveled by a car at time, then the first derivative $f'(t)$ gives the rate of change of distance traveled as a function of time, i.e., its speed. Then the second derivative $f''(t)$ gives the rate of change of the speed of the car, i.e., its acceleration.
  - First derivatives are useful for identifying stationary points, values of x or t or any other input at which point the function either stops increasing or stops decreasing. For our purposes, setting first derivatives equal to zero—finding first-order conditions—will be how we identify minimum and maximum points.
  - Second derivatives are useful for characterizing stationary points. The first-order condition only tells us where the first derivative is equal to zero, but it does not tell us whether those points represent a local minimum or a local maximum. The second derivative does this for us: if the second derivative is positive at a stationary point, then we know it is a local minimum. If the second derivative is negative at a stationary point, then we know it is a local maximum.

- Partial derivatives
  - When f is a function of multiple inputs or variables, then differentiation entails calculating partial derivatives with respect to each variable, holding the other ones constant. For example, suppose our function of interest is $f(x_1, x_2) = x^2 + \ln(4x_2)$. Then we can take partial derivatives with respect to either x_1 or x_2 as follows:

$$
\begin{aligned}
\frac{\partial}{\partial x_1} f(x_1, x_2) &= \frac{\partial}{\partial x_1}(x_1^2 \ln(4x_2)) \\
&= 2x_1 \ln(4x_2)) \\
\frac{\partial}{\partial x_1} f(x_1, x_2) &= \frac{\partial}{\partial x_2}(x_1^2 \ln(4x_2)) \\
&= 4x_1^2 \frac{1}{4x_2} = \frac{x_1^2}{x_2}
\end{aligned} \tag{8}
$$

  - Higher-order partial derivatives work analogously to the single-variable case other than we can take the partial derivative a multivariate function with respect to one variable, then take the partial derivative of that partial derivative with respect to another:

$$
\begin{aligned}
f(x, y) &= 4x^3y^2 \\
\frac{\partial}{\partial x} f(x, y) &= 12x^2y^2 \\
\frac{\partial^2}{\partial x \partial y} f(x, y) &= 24x^2y \\
\frac{\partial}{\partial y} f(x, y) &= 8x^3y \\
\frac{\partial^2}{\partial y \partial x} f(x, y) &= 24x^2y
\end{aligned} \tag{9}
$$

  - Note it is an identity that $\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial^2}{\partial y \partial x} f(x, y)$. That is to say, it is always true by definition.

4 Unconstrained optimization

- An optimization problem generally has the following components:
  1. The **objective function**: the function the economic agent wants to optimize. For example, a consumer might want to maximize their utility or a producer might want to minimize their costs:

$$
\max U(x_1, x_2) \tag{10}
$$

2. The **choice variables**: this is the set of input variables that the economic agent is able to choose values of in order to optimize their objective function. A consumer seeking to maximize their utility may be doing so **with respect to** consumption of goods x_1 and x_2. The optimization problem is then to maximize utility with respect to x_1 and x_2:

$$\max_{\{x_1, x_1\}} U(x_1, x_2) \tag{11}$$

3. The **constraints**, if they exist, place restrictions on the values the choice variables can take. For example, in a standard model where consuming more goods always gives more utility, we need restrictions on how many goods can be consumed in order for the optimization problem to be economically interesting. These often take the form of budget constraints where the consumer has a maximum budget B and the goods are priced p_1 and p_2. Then the optimization problem becomes

$$\begin{aligned} &\max_{\{x_1, x_1\}} U(x_1, x_2) \\ &\text{s.t. } p_1 x_1 + p_2 x_2 \leq B \end{aligned} \tag{12}$$

where s.t. can be read as "subject to" or "such that"

- We'll be primarily concerned with constrained optimization problems throughout the semester, but let's limit today's discussion to the unconstrained case using objective functions that are non-monotonic

- Solving an optimization problem usually involves the following steps:

1. **Writing the problem:** identifying the objective function, the type of optimization we want (maximization or minimization), the choice variables, and if applicable, the constraints

2. **Taking the first-order condition** with respect to each choice variable: set the first-order conditions equal to zero and solve for the relevant choice variable

3. **Checking the second-order conditions**: verify through the second derivative that the stationary point identified by the first-order condition is of the type we want (a maximum or minimum if our objective is to maximize or minimize the objective function)

- An example in one dimension: A consumer seeks to maximize their utility from consumption of good x. If their utility function is given by $u(x) = -(x-2)^2$, what is the optimal amount of x for the consumer?

1. Writing the problem:

$$\max_x -(x-2)^2 \tag{13}$$

2. Taking the first-order condition:

$$\begin{aligned} 0 &= \frac{d}{dx}\left(-(x-2)^2\right) \\ &= -2(x-2) \\ \Rightarrow x &= 2 \end{aligned} \tag{14}$$

3. Checking the second-order condition:

$$\begin{aligned} \frac{d^2}{dx^2}\left(-(x-2)^2\right) &= \frac{d}{dx}\left(-2(x-2)\right) \\ &= -2 \end{aligned} \tag{15}$$

This is negative when $x = 2$ and so the stationary point identified by the first-order condition ($x = 2$) must be a local maximum, just as we wanted

- When we are optimizing in multiple dimensions, we'll have to take the first-order condition with respect to all choice variables. This entails taking partial derivatives and setting them equal to 0 with respect to each choice variable and then solving for the choice variables. Each first-order condition gives a system of equations which can then be solved jointly. So for example, imagine the utility function in the previous example was instead $u(x_1, x_2) = -3(x_2-1)^2 - x_1^2$

1. Writing the problem:

$$\max_{\{x_1, x_2\}} -3(x_2-1)^2 - x_1^2 \tag{16}$$

2. Taking first-order conditions:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_1}\left[-3(x_2-1)^2 - x_1^2\right] \\ &= -2x_1 \\ \Rightarrow x_1^* &= 0 \\ 0 &= \frac{\partial}{\partial x_2}\left[-3(x_2-1)^2 - x_1^2\right] \\ &= -6(x_2-1) \\ \Rightarrow x_2^* &= 1 \end{aligned} \tag{17}$$

3. We can analogously verify that the second derivatives of the objective function with respect to the two choice variables satisfy the condition for the bundle (x_1^*, x_2^*) to be a local maximum